the density of a liquid, we are referring to its macroscopic density. We regard it as a continuous fluid and ignore its discrete molecular constitution.

The field due to a continuous charge distribution can be obtained in much the same way as for a system of discrete charges, Eq. (1.10). Suppose a continuous charge distribution in space has a charge density ρ. Choose any convenient origin O and let the position vector of any point in the charge distribution be **r**. The charge density ρ may vary from point to point, i.e., it is a function of **r**. Divide the charge distribution into small volume elements of size ΔV. The charge in a volume element ΔV is $\rho \Delta V$.

Now, consider any general point P (inside or outside the distribution) with position vector **R** (Fig. 1.24). Electric field due to the charge $\rho \Delta V$ is given by Coulomb's law:

$$\Delta \mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{\rho \, \Delta V}{r'^2} \hat{\mathbf{r}}' \tag{1.26}$$

where r' is the distance between the charge element and P, and $\hat{\mathbf{r}}'$ is a unit vector in the direction from the charge element to P. By the superposition principle, the total electric field due to the charge distribution is obtained by summing over electric fields due to different volume elements:

$$\mathbf{E} \cong \frac{1}{4\pi\varepsilon_0} \sum_{all\ \Delta V} \frac{\rho \, \Delta V}{r'^2} \hat{\mathbf{r}}' \tag{1.27}$$

Note that ρ, r', $\hat{\mathbf{r}}'$ all can vary from point to point. In a strict mathematical method, we should let $\Delta V \rightarrow 0$ and the sum then becomes an integral; but we omit that discussion here, for simplicity. In short, using Coulomb's law and the superposition principle, electric field can be determined for any charge distribution, discrete or continuous or part discrete and part continuous.

1.14 Gauss's Law

As a simple application of the notion of electric flux, let us consider the total flux through a sphere of radius r, which encloses a point charge q at its centre. Divide the sphere into small area elements, as shown in Fig. 1.25.

The flux through an area element $\Delta \mathbf{S}$ is

$$\Delta \phi = \mathbf{E} \cdot \Delta \mathbf{S} = \frac{q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}} \cdot \Delta \mathbf{S} \tag{1.28}$$

where we have used Coulomb's law for the electric field due to a single charge q. The unit vector $\hat{\mathbf{r}}$ is along the radius vector from the centre to the area element. Now, since the normal to a sphere at every point is along the radius vector at that point, the area element $\Delta \mathbf{S}$ and $\hat{\mathbf{r}}$ have the same direction. Therefore,

$$\Delta \phi = \frac{q}{4\pi\varepsilon_0 r^2} \Delta S \tag{1.29}$$

since the magnitude of a unit vector is 1.

The total flux through the sphere is obtained by adding up flux through all the different area elements:

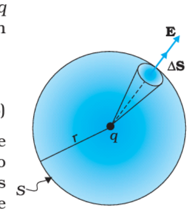

FIGURE 1.25 Flux through a sphere enclosing a point charge q at its centre.

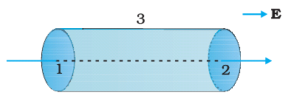

FIGURE 1.26 Calculation of the flux of uniform electric field through the surface of a cylinder.

$$\phi = \sum_{\text{all } \Delta S} \frac{q}{4\pi\varepsilon_0 r^2} \Delta S$$

Since each area element of the sphere is at the same distance r from the charge,

$$\phi = \frac{q}{4\pi\varepsilon_o r^2} \sum_{\text{all } \Delta S} \Delta S = \frac{q}{4\pi\varepsilon_0 r^2} S$$

Now S, the total area of the sphere, equals $4\pi r^2$. Thus,

$$\phi = \frac{q}{4\pi\varepsilon_0 r^2} \times 4\pi r^2 = \frac{q}{\varepsilon_0} \qquad (1.30)$$

Equation (1.30) is a simple illustration of a general result of electrostatics called Gauss's law.

We state *Gauss's law* without proof:

Electric flux through a closed surface S

$= q/\varepsilon_0$ (1.31)

q = total charge enclosed by S.

The law implies that the total electric flux through a closed surface is zero if no charge is enclosed by the surface. We can see that explicitly in the simple situation of Fig. 1.26.

Here the electric field is uniform and we are considering a closed cylindrical surface, with its axis parallel to the uniform field **E**. The total flux ϕ through the surface is $\phi = \phi_1 + \phi_2 + \phi_3$, where ϕ_1 and ϕ_2 represent the flux through the surfaces 1 and 2 (of circular cross-section) of the cylinder and ϕ_3 is the flux through the curved cylindrical part of the closed surface. Now the normal to the surface 3 at every point is perpendicular to **E**, so by definition of flux, $\phi_3 = 0$. Further, the outward normal to 2 is along **E** while the outward normal to 1 is opposite to **E**. Therefore,

$$\phi_1 = -E\,S_1, \qquad \phi_2 = +E\,S_2$$
$$S_1 = S_2 = S$$

where S is the area of circular cross-section. Thus, the total flux is zero, as expected by Gauss's law. Thus, whenever you find that the net electric flux through a closed surface is zero, we conclude that the total charge contained in the closed surface is zero.

The great significance of Gauss's law Eq. (1.31), is that it is true in general, and not only for the simple cases we have considered above. Let us note some important points regarding this law:

(i) Gauss's law is true for any closed surface, no matter what its shape or size.

(ii) The term q on the right side of Gauss's law, Eq. (1.31), includes the sum of all charges enclosed by the surface. The charges may be located anywhere inside the surface.

(iii) In the situation when the surface is so chosen that there are some charges inside and some outside, the electric field [whose flux appears on the left side of Eq. (1.31)] is due to all the charges, both inside and outside S. The term q on the right side of Gauss's law, however, represents only the total charge inside S.

(iv) The surface that we choose for the application of Gauss's law is called the Gaussian surface. You may choose any Gaussian surface and apply Gauss's law. However, take care not to let the Gaussian surface pass through any discrete charge. This is because electric field due to a system of discrete charges is not well defined at the location of any charge. (As you go close to the charge, the field grows without any bound.) However, the Gaussian surface can pass through a continuous charge distribution.

(v) Gauss's law is often useful towards a much easier calculation of the electrostatic field *when the system has some symmetry*. This is facilitated by the choice of a suitable Gaussian surface.

(vi) Finally, Gauss's law is based on the inverse square dependence on distance contained in the Coulomb's law. Any violation of Gauss's law will indicate departure from the inverse square law.

Example 1.11 The electric field components in Fig. 1.27 are $E_x = \alpha x^{1/2}$, $E_y = E_z = 0$, in which α = 800 N/C m$^{1/2}$. Calculate (a) the flux through the cube, and (b) the charge within the cube. Assume that a = 0.1 m.

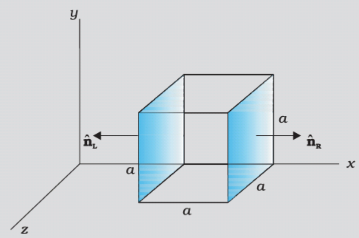

FIGURE 1.27

Solution

(a) Since the electric field has only an x component, for faces perpendicular to x direction, the angle between **E** and Δ**S** is $\pm \pi/2$. Therefore, the flux $\phi = \mathbf{E}.\Delta\mathbf{S}$ is separately zero for each face of the cube except the two shaded ones. Now the magnitude of the electric field at the left face is

$E_L = \alpha x^{1/2} = \alpha a^{1/2}$

($x = a$ at the left face).

The magnitude of electric field at the right face is

$E_R = \alpha\, x^{1/2} = \alpha\, (2a)^{1/2}$

($x = 2a$ at the right face).

The corresponding fluxes are

$\phi_L = \mathbf{E}_L\cdot\Delta\mathbf{S} = \Delta S\, \mathbf{E}_L \cdot \hat{\mathbf{n}}_L = E_L\, \Delta S \cos\theta = -E_L\, \Delta S$, since $\theta = 180°$

$= -E_L a^2$

$\phi_R = \mathbf{E}_R\cdot\Delta\mathbf{S} = E_R\, \Delta S \cos\theta = E_R\, \Delta S$, since $\theta = 0°$

$= E_R a^2$

Net flux through the cube

EXAMPLE 1.11

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Screenshot is done
Tap to view

$= \phi_R + \phi_L = E_R a^2 - E_L a^2 = a^2 (E_R - E_L) = \alpha a^2 [(2a)^{1/2} - a^{1/2}]$

$= \alpha a^{5/2} \left(\sqrt{2} - 1\right)$

$= 800\ (0.1)^{5/2} \left(\sqrt{2} - 1\right)$

$= 1.05\ \text{N m}^2\ \text{C}^{-1}$

(b) We can use Gauss's law to find the total charge q inside the cube. We have $\phi = q/\varepsilon_0$ or $q = \phi\varepsilon_0$. Therefore,

$q = 1.05 \times 8.854 \times 10^{-12}\ \text{C} = 9.27 \times 10^{-12}\ \text{C}.$

EXAMPLE 1.11

Example 1.12 An electric field is uniform, and in the positive x direction for positive x, and uniform with the same magnitude but in the negative x direction for negative x. It is given that $\mathbf{E} = 200\ \hat{\mathbf{i}}$ N/C for $x > 0$ and $\mathbf{E} = -200\ \hat{\mathbf{i}}$ N/C for $x < 0$. A right circular cylinder of length 20 cm and radius 5 cm has its centre at the origin and its axis along the x-axis so that one face is at $x = +10$ cm and the other is at $x = -10$ cm (Fig. 1.28). (a) What is the net outward flux through each flat face? (b) What is the flux through the side of the cylinder? (c) What is the net outward flux through the cylinder? (d) What is the net charge inside the cylinder?

Solution

(a) We can see from the figure that on the left face $\mathbf{E}$ and $\Delta\mathbf{S}$ are parallel. Therefore, the outward flux is

$\phi_L = \mathbf{E} \cdot \Delta\mathbf{S} = -200\ \hat{\mathbf{i}} \cdot \Delta\mathbf{S}$

$= +200\ \Delta S$, since $\hat{\mathbf{i}} \cdot \Delta\mathbf{S} = -\Delta S$

$= +200 \times \pi\ (0.05)^2 = +1.57\ \text{N m}^2\ \text{C}^{-1}$

On the right face, $\mathbf{E}$ and $\Delta\mathbf{S}$ are parallel and therefore

$\phi_R = \mathbf{E} \cdot \Delta\mathbf{S} = +1.57\ \text{N m}^2\ \text{C}^{-1}$.

(b) For any point on the side of the cylinder $\mathbf{E}$ is perpendicular to $\Delta\mathbf{S}$ and hence $\mathbf{E} \cdot \Delta\mathbf{S} = 0$. Therefore, the flux out of the side of the cylinder is zero.

(c) Net outward flux through the cylinder

$\phi = 1.57 + 1.57 + 0 = 3.14\ \text{N m}^2\ \text{C}^{-1}$

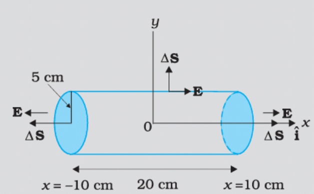


FIGURE 1.28

(d) The net charge within the cylinder can be found by using Gauss's law which gives

$q = \varepsilon_0 \phi$

$= 3.14 \times 8.854 \times 10^{-12}\ \text{C}$

$= 2.78 \times 10^{-11}\ \text{C}$

EXAMPLE 1.12

1.15 Applications of Gauss's Law

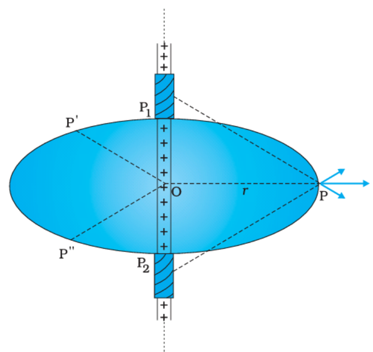
The electric field due to a general charge distribution is, as seen above, given by Eq. (1.27). In practice, except for some special cases, the summation (or integration) involved in this equation cannot be carried out to give electric field at every point in space. For some symmetric charge configurations, however, it is possible to obtain the electric field in a simple way using the Gauss's law. This is best understood by some examples.

1.15.1 Field due to an infinitely long straight uniformly charged wire

Consider an infinitely long thin straight wire with uniform linear charge density λ. The wire is obviously an axis of symmetry. Suppose we take the radial vector from O to P and rotate it around the wire. The points P, P′, P″ so obtained are completely equivalent with respect to the charged wire. This implies that the electric field must have the same magnitude at these points. The direction of electric field at every point must be radial (outward if $\lambda > 0$, inward if $\lambda < 0$). This is clear from Fig. 1.29.

Consider a pair of line elements P_1 and P_2 of the wire, as shown. The electric fields produced by the two elements of the pair when summed give a resultant electric field which is radial (the components normal to the radial vector cancel). This is true for any such pair and hence the total field at any point P is radial. Finally, since the wire is infinite, electric field does not depend on the position of P along the length of the wire. In short, the electric field is everywhere radial in the plane cutting the wire normally, and its magnitude depends only on the radial distance r.

To calculate the field, imagine a cylindrical Gaussian surface, as shown in the Fig. 1.29(b). Since the field is everywhere radial, flux through the two ends of the cylindrical Gaussian surface is zero. At the cylindrical part of the surface, **E** is normal to the surface at every point, and its magnitude is constant, since it depends only on r. The surface area of the curved part is $2\pi rl$, where l is the length of the cylinder.



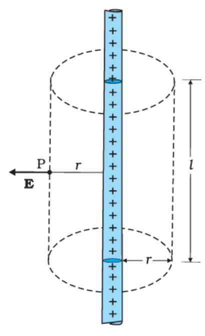


FIGURE 1.29 (a) Electric field due to an infinitely long thin straight wire is radial, (b) The Gaussian surface for a long thin wire of uniform linear charge density.

Flux through the Gaussian surface

= flux through the curved cylindrical part of the surface

= $E \times 2\pi rl$

The surface includes charge equal to λl. Gauss's law then gives
$E \times 2\pi rl = \lambda l/\varepsilon_0$

i.e., $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

Vectorially, **E** at any point is given by

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{\mathbf{n}} \qquad (1.32)$$

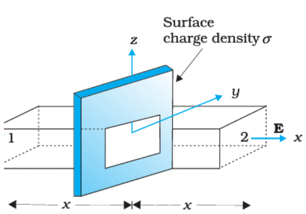
where $\hat{\mathbf{n}}$ is the radial unit vector in the plane normal to the wire passing through the point. **E** is directed outward if λ is positive and inward if λ is negative.

Note that when we write a vector **A** as a scalar multiplied by a unit vector, i.e., as $\mathbf{A} = A\,\hat{\mathbf{a}}$, the scalar A is an algebraic number. It can be negative or positive. The direction of **A** will be the same as that of the unit vector $\hat{\mathbf{a}}$ if $A > 0$ and opposite to $\hat{\mathbf{a}}$ if $A < 0$. When we want to restrict to non-negative values, we use the symbol $|\mathbf{A}|$ and call it the modulus of **A**. Thus, $|\mathbf{A}| \geq 0$.

Also note that though only the charge enclosed by the surface (λl) was included above, the electric field **E** is due to the charge on the entire wire. Further, the assumption that the wire is infinitely long is crucial. Without this assumption, we cannot take **E** to be normal to the curved part of the cylindrical Gaussian surface. However, Eq. (1.32) is approximately true for electric field around the central portions of a long wire, where the end effects may be ignored.

1.15.2 Field due to a uniformly charged infinite plane sheet

Let σ be the uniform surface charge density of an infinite plane sheet (Fig. 1.30). We take the x-axis normal to the given plane. By symmetry, the electric field will not depend on y and z coordinates and its direction at every point must be parallel to the x-direction.



URE 1.30 Gaussian surface for a ormly charged infinite plane sheet.

We can take the Gaussian surface to be a rectangular parallelepiped of cross sectional area A, as shown. (A cylindrical surface will also do.) As seen from the figure, only the two faces 1 and 2 will contribute to the flux; electric field lines are parallel to the other faces and they, therefore, do not contribute to the total flux.

The unit vector normal to surface 1 is in $-x$ direction while the unit vector normal to surface 2 is in the $+x$ direction. Therefore, flux $\mathbf{E}.\Delta\mathbf{S}$ through both the surfaces are equal and add up. Therefore the net flux through the Gaussian surface is $2\,EA$. The charge enclosed by the closed surface is σA. Therefore by Gauss's law,

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$2\,EA = \sigma A/\varepsilon_0$
or, $E = \sigma/2\varepsilon_0$
Vectorically,

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}} \tag{1.33}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane and going away from it.

E is directed away from the plate if σ is positive and toward the plate if σ is negative. Note that the above application of the Gauss' law has brought out an additional fact: E is independent of x also.

For a finite large planar sheet, Eq. (1.33) is approximately true in the middle regions of the planar sheet, away from the ends.

1.15.3 Field due to a uniformly charged thin spherical shell

Let σ be the uniform surface charge density of a thin spherical shell of radius R (Fig. 1.31). The situation has obvious spherical symmetry. The field at any point P, outside or inside, can depend only on r (the radial distance from the centre of the shell to the point) and must be radial (i.e., along the radius vector).

(i) *Field outside the shell:* Consider a point P outside the shell with radius vector **r**. To calculate **E** at P, we take the Gaussian surface to be a sphere of radius r and with centre O, passing through P. All points on this sphere are equivalent relative to the given charged configuration. (That is what we mean by spherical symmetry.) The electric field at each point of the Gaussian surface, therefore, has the same magnitude E and is along the radius vector at each point. Thus, **E** and $\Delta\mathbf{S}$ at every point are parallel and the flux through each element is $E\,\Delta S$. Summing over all ΔS, the flux through the Gaussian surface is $E \times 4\pi r^2$. The charge enclosed is $\sigma \times 4\pi R^2$. By Gauss's law

$$E \times 4\pi r^2 = \frac{\sigma}{\varepsilon_0} 4\pi R^2$$

$$\text{Or,}\quad E = \frac{\sigma R^2}{\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^2}$$

where $q = 4\pi R^2 \sigma$ is the total charge on the spherical shell.
Vectorially,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \tag{1.34}$$

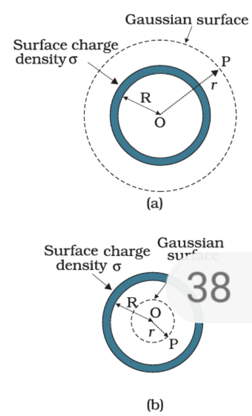

FIGURE 1.31 Gaussian surfaces for a point with (a) $r > R$, (b) $r < R$.

The electric field is directed outward if $q > 0$ and inward if $q < 0$. This, however, is exactly the field produced by a charge q placed at the centre O. Thus for points outside the shell, the field due to a uniformly charged shell is as if the entire charge of the shell is concentrated at its centre.

(ii) *Field inside the shell:* In Fig. 1.31(b), the point P is inside the shell. The Gaussian surface is again a sphere through P centred at O.

centrated at its centre.

Field inside the shell: In Fig. 1.31(b), the point P is inside the shell. The Gaussian surface is again a sphere through P centred at O.

The flux through the Gaussian surface, calculated as before, is $E \times 4\,\pi\,r^2$. However, in this case, the Gaussian surface encloses no charge. Gauss's law then gives

$$E \times 4\,\pi\,r^2 = 0$$

i.e., $E = 0 \qquad (r < R)$ (1.35)

that is, the field due to a uniformly charged thin shell is zero at all points inside the shell*. This important result is a direct consequence of Gauss's law which follows from Coulomb's law. The experimental verification of this result confirms the $1/r^2$ dependence in Coulomb's law.

Example 1.13 An early model for an atom considered it to have a positively charged point nucleus of charge *Ze*, surrounded by a uniform density of negative charge up to a radius *R*. The atom as a whole is neutral. For this model, what is the electric field at a distance *r* from the nucleus?

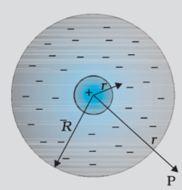

FIGURE 1.32

Solution The charge distribution for this model of the atom is as shown in Fig. 1.32. The total negative charge in the uniform spherical charge distribution of radius *R* must be $-Z\,e$, since the atom (nucleus of charge $Z\,e$ + negative charge) is neutral. This immediately gives us the negative charge density ρ, since we must have

$$\frac{4\pi R^3}{3}\rho = 0 - Ze$$

or $\rho = -\dfrac{3\,Ze}{4\,\pi\,R^3}$

To find the electric field **E(r)** at a point P which is a distance *r* away from the nucleus, we use Gauss's law. Because of the spherical symmetry of the charge distribution, the magnitude of the electric field **E(r)** depends only on the radial distance, no matter what the direction of **r**. Its direction is along (or opposite to) the radius vector **r** from the origin to the point P. The obvious Gaussian surface is a spherical surface centred at the nucleus. We consider two situations, namely, $r < R$ and $r > R$.

(i) $r < R$: The electric flux ϕ enclosed by the spherical surface is

$$\phi = E\,(r) \times 4\,\pi\,r^2$$

where $E\,(r)$ is the magnitude of the electric field at *r*. This i

EXAMPLE 1.13

* Compare this with a uniform mass shell discussed in Sectio Textbook of Physics.

the field at any point on the spherical Gaussian surface has the same direction as the normal to the surface there, and has the same magnitude at all points on the surface.

The charge q enclosed by the Gaussian surface is the positive nuclear charge and the negative charge within the sphere of radius r,

i.e., $q = Ze + \dfrac{4\pi r^3}{3}\rho$

Substituting for the charge density ρ obtained earlier, we have

$$q = Ze - Ze\frac{r^3}{R^3}$$

Gauss's law then gives,

$$E(r) = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right); \quad r < R$$

The electric field is directed radially outward.

(ii) $r > R$: In this case, the total charge enclosed by the Gaussian spherical surface is zero since the atom is neutral. Thus, from Gauss's law,

$E(r) \times 4\pi r^2 = 0$ or $E(r) = 0$; $\quad r > R$

At $r = R$, both cases give the same result: $E = 0$.

EXAMPLE 1.13

ON SYMMETRY OPERATIONS

In Physics, we often encounter systems with various symmetries. Consideration of these symmetries helps one arrive at results much faster than otherwise by a straightforward calculation. Consider, for example an infinite uniform sheet of charge (surface charge density σ) along the y-z plane. This system is unchanged if (a) translated parallel to the y-z plane in any direction, (b) rotated about the x-axis through any angle. As the system is unchanged under such symmetry operation, so must its properties be. In particular, in this example, the electric field **E** must be unchanged.

Translation symmetry along the y-axis shows that the electric field must be the same at a point $(0, y_1, 0)$ as at $(0, y_2, 0)$. Similarly translational symmetry along the z-axis shows that the electric field at two point $(0, 0, z_1)$ and $(0, 0, z_2)$ must be the same. By using rotation symmetry around the x-axis, we can conclude that **E** must be perpendicular to the y-z plane, that is, it must be parallel to the x-direction.

Try to think of a symmetry now which will tell you that the magnitude of the electric field is a constant, independent of the x-coordinate. It thus turns out that the magnitude of the electric field due to a uniformly charged infinite conducting sheet is the same at all points in space. The direction, however, is opposite of each other on either side of the sheet.

Compare this with the effort needed to arrive at this result by a direct calculation using Coulomb's law.